X : Calabi-Yau 3-fold $\qquad K_X = 0$

projective $\longrightarrow$ later proj. over affine

· Gromov-Witten invariants

· Donaldson-Thomas (DT) invariants

- counting ideal sheaves $\mathcal{I}_Z \subset \mathcal{O}_X$ of 1 dim. subscheme Z

$$I_{n,\beta} := \int_{[M_X(n,\beta)]^{vir.}} 1 \quad \in \mathbb{Z}$$

$\chi(\mathcal{O}_Z) = n$

$[Z] = \beta \in H_2(X, \mathbb{Z})$

- moduli space of ideals
- perfect obstruction theory
- virtual dim. $= 0$

$$\sum_i (-1)^i \dim \mathrm{Ext}^i (I_Z, I_Z) = 0$$

· Pandharipande-Thomas (PT) invariants

- counting pairs $\quad \mathcal{O}_X \xrightarrow{s} F$
  - F is pure of dim 1
  - s has 0-dim. cokernel

Rem. This is a variant of an ideal sheaf :

$\mathcal{O}_X \xrightarrow{s} \mathcal{O}_Z$ - $\mathcal{O}_Z$ may contains 0-dim subsheaf : $\mathcal{O}_{Z_0}$

- surjective

The change can be understood as "wall-crossing"

$$\underset{\text{sub}}{\mathcal{O}_{Z_0} \subset \mathcal{O}_Z} \quad \rightsquigarrow \quad \underset{\text{quotient}}{F \twoheadrightarrow \mathcal{O}_{Z_0}}$$

X : resolved conifold $=$ total space of $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$

$\longrightarrow X_0 = \{xy = zw\} \subset \mathbb{C}^4$

- Szendrői : noncommutative DT invariant

count "perverse ideal sheaves"

$\cdots$ objects in $D^b(\mathrm{Coh}\,X)$

$D^b(\mathrm{Coh}\,X)$: derived category of coherent sheaves

$\cup$

$\mathrm{Per}(X)$: abelian category ($=$ heart of t-structure)

$\neq \mathrm{Coh}(X)$

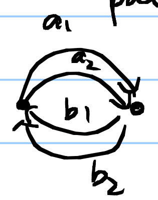
Bridgeland, Van den Bergh

$D^b_c(\mathrm{Coh}\,X) \cong D^b(\mathrm{mod}\,A)$

cpt supports

finite dim. repr.

path alg. with relation



$a_1 b_i a_2 = a_2 b_i a_1$

$b_1 a_i b_2 = b_2 a_i b_1$

$(i = 1, 2)$

$\mathrm{Per}_c(X) \cong \mathrm{mod}\,A$

$$Z_{PT}(q,t) = \prod_{m=1}^{\infty} (1-(-q)^m t)^m$$

$$Z_{DT}(q,t) = \left\{ \prod_{m=1}^{\infty} (1-(-q)^m)^{-m} \right\}^2 \prod_{m=1}^{\infty} (1-(-q)^m t)^m$$

↑ contribution from 0-dim. subscheme

$$Z_{NCDT}(q,t) = \left\{ \prod_{m=1}^{\infty} (1-(-q)^m)^{-m} \right\}^2 \prod_{m=1}^{\infty} (1-(-q)^m t)^m \times \prod_{m=1}^{\infty} (1-(-q)^m t^{-1})^m$$

<u>Rem</u>. $X \dashrightarrow X^+$ flop replace t by t^{-1}

NCDT contains "contributions"

both curves in X & X^+

joint work with Kentaro Nagao:

Introduce more variants depending on the stability parameters $\zeta = (\zeta_0, \zeta_1)$

<u>Def.</u> A system $(\mathcal{O}_X \xrightarrow{s} F)$ is ζ-stable

↑ perverse sheaf

$\Leftrightarrow$. $\forall E \subset F$ subobject

$\zeta_0 \dim H^0(E) + \zeta_1 \dim H^0(E \otimes \mathcal{L}) \leq 0$

. $\forall E \subset F$ subobject s.t. s factors

$\zeta_0 \dim H^0(E) + \zeta_1 \dim H^0(E \otimes \mathcal{L})$

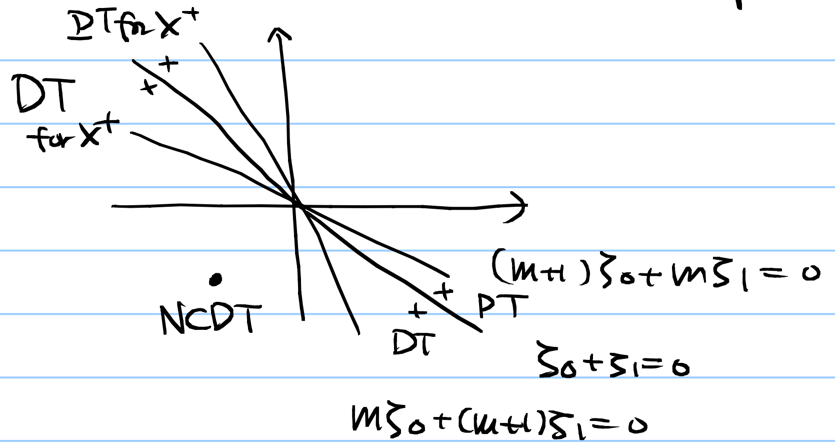
$\leq \zeta_0 \dim H^0(F) + \zeta_1 \dim H^0(F \otimes \mathcal{L})$

$\mathcal{L}$ = pull back of $\mathcal{O}_{\mathbb{P}^1}(1)$

$\rightsquigarrow$ Can define invariants

<u>Rem.</u> stability comes from GIT construction of moduli space of rep. of quiver A

<u>Th</u> (1) $\exists$ chamber structure on the space of $\zeta \in \mathbb{R}^2$



$(m \in \mathbb{Z}_{\geq 0})$

inv. depend only on the chamber.

(2) In certain chambers invariants reproduces NCDT, DT, PT for X & X^+

(3) inv.'s have ∞-product formulas :

$$Z_{NCDT}(q,t) = \left\{ \prod_{m=1}^{\infty} (1-(-q)^m)^{-m} \right\}^2 \underbrace{\prod_{m=1}^{\infty} (1-(-q)^m t)^m}$$

$$\times \underbrace{\prod_{m=1}^{\infty} (1-(-q)^m t^{-1})^m}$$

starting from NCDT, we loose a factor of

or

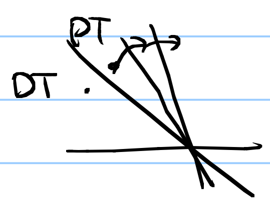
one by one when we cross a wall from $m=1$ to ∞.

$\rightsquigarrow$ eventually we loose all or and arrive DT for Y or Y^+.

Next we cross $\zeta_0+\zeta_1=0$.

Then we loose $\left\{\prod_{m=1}^{\infty}(1-(-q)^m)^{-m}\right\}^2$ to get PT inv.

Then



we further cross the wall

we loose factors in or .

Finally

inv. $= \underline{1}$

Rem. ∘ Original proof of the formula for NCDT (due to B. Young) is combinatorial.
We have the proof via wall-crossing

∘ Nagao generalize this result

$$\begin{array}{ccc} X & \longrightarrow & X_0 \\ \text{↖ toric CY} & & \uparrow \text{ small contraction} \\ & & \text{affine} \end{array}$$

∘ The wall-crossing formula is an example of more general formula by Joyce, Kontsevich-Soibelman.